

Canonical supergravity with Barbero-Immirzi parameter

Sandipan Sengupta^{1,*} and Romesh K. Kaul^{1,†}

¹*The Institute of Mathematical Sciences*

CIT Campus, Chennai-600 113, INDIA.

Abstract

A canonical formulation of the $N = 1$ supergravity theory containing the topological Nieh-Yan term in its Lagrangian density is developed. The constraints are analysed without choosing any gauge. In the time gauge, the theory is shown to be described in terms of real $SU(2)$ variables.

PACS numbers: 04.20.Fy, 04.60.-m, 04.60.Ds, 04.60.Pp

*Electronic address: sandi@imsc.res.in

†Electronic address: kaul@imsc.res.in

I. INTRODUCTION

Ashtekar's canonical formulation of gravity in terms of complex Yang-Mills connection variables has provided a gauge theoretic interpretation of gravity[1]. Subsequently, Barbero and Immirzi have reframed this description in terms of real $SU(2)$ variables[2]. These variables have been shown to originate from the Holst Lagrangian density[3], which is written in the first order form with tetrads (e) and spin connections (ω) as independent variables :

$$\mathcal{L} = \frac{1}{2}e\Sigma_{IJ}^{\mu\nu}R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2}e\Sigma_{IJ}^{\mu\nu}\tilde{R}_{\mu\nu}{}^{IJ}(\omega) \quad (1)$$

where,

$$\Sigma_{IJ}^{\mu\nu} := \frac{1}{2}(e_I^\mu e_J^\nu - e_J^\mu e_I^\nu), \quad R_{\mu\nu}{}^{IJ}(\omega) := \partial_{[\mu}\omega_{\nu]}{}^{IJ} + \omega_{[\mu}{}^{IK}\omega_{\nu]}{}_K{}^J, \quad \tilde{R}_{\mu\nu}{}^{IJ}(\omega) := \frac{1}{2}\epsilon^{IJKL}R_{\mu\nu KL}(\omega).$$

Here η , the inverse of the Barbero-Immirzi parameter, is the coefficient of the Holst term. This additional term preserves the classical equations of motion given by the Hilbert-Palatini action. Thus η appears as a free parameter in this framework. Hamiltonian analysis of this theory based on the Lagrangian density (1) has been presented in ref.[3, 4].

When matter is coupled to pure gravity, one needs additional terms apart from the Holst term so that the equations of motion continue to be independent of η . Actions containing such modifications have been found for a few cases, e.g., spin- $\frac{1}{2}$ fermions and $N = 1, 2, 4$ supergravity theories [5, 6]. A superspace formalism for $N = 1$ supergravity has been presented in [7], which reproduces the result of [6] for this theory. It has also been noted in ref.[6] that although these modifications of the Holst term for different matter couplings follow a generic pattern in the sense that they can be written as a total divergence after using the connection equation of motion (see [5] also), they are not universal. To emphasise, the modified Holst terms needed to preserve the equations of motion change with the matter content of the theory.

A universal prescription for finding a generalised action leading to a real $SU(2)$ formulation of gravity with or without matter was proposed in [8]. This involves a Lagrangian density containing a topological term in the form of the Nieh-Yan invariant [9] instead of the original Holst term:

$$\mathcal{L} = \frac{1}{2}e\Sigma_{IJ}^{\mu\nu}R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2}I_{NY} \quad (2)$$

where

$$I_{NY} = \epsilon^{\mu\nu\alpha\beta} \left[D_\mu(\omega) e_\nu^I D_\alpha(\omega) e_{I\beta} - \frac{1}{2} \Sigma_{\mu\nu}^{IJ} R_{\alpha\beta IJ}(\omega) \right] , \quad D_\mu(\omega) e_\nu^I := \partial_\mu e_\nu^I + \omega_\mu^I{}_J e_\nu^J . \quad (3)$$

I_{NY} is a total divergence, given by:

$$I_{NY} = \partial_\mu (\epsilon^{\mu\nu\alpha\beta} e_\nu^I D_\alpha e_{\beta I})$$

Thus it does not affect the classical equations of motion, even when matter is coupled to the Lagrangian (2).

The action in (2) brings with it the crucial feature that η can be provided a topological interpretation in any theory of gravity with or without matter. This is in contrast to the Holst action where η is not a coefficient of a topological term.

The Nieh-Yan term, being a topological density, can be written uniquely in terms of the geometric variables (e, ω) . Thus one does not need to look for a new ‘modified Holst term’ whenever the matter content changes, unlike the earlier approaches which were matter-specific. As an elucidation of this fact, this method has been applied to spin- $\frac{1}{2}$ fermions in ref.[8].

Here in this brief report we analyse the case of $N = 1$ supergravity. The canonical treatment of this theory has been considered earlier in several contexts [10, 11]. In ref.[11], the Hamiltonian analysis of the corresponding Holst action has been carried out in time gauge. Here we consider a Lagrangian density describing the same theory, but containing the Nieh-Yan invariant instead of the Holst term in addition to the usual Hilbert-Palatini and spin- $\frac{3}{2}$ fermionic terms. In the next section, we exhibit the canonical formulation of this action, closely following the analysis as given in [4, 8]. Then we demonstrate that the set of constraints in the time gauge leads to a real $SU(2)$ description of this theory in terms of the Barbero-Immirzi connection. We also add a few comments on how to recover the correct transformation properties of the fields under the action of the symmetry generators.

II. N=1 SUPERGRAVITY

The Lagrangian density for gravity coupled to spin- $\frac{3}{2}$ Majorana fermions is given by [12]:

$$\mathcal{L} = \frac{1}{2} e \Sigma_{IJ}^{\mu\nu} R_{\mu\nu}{}^{IJ}(\omega) + \frac{i}{2} \epsilon^{\mu\nu\alpha\beta} \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\alpha(\omega) \psi_\beta \quad (4)$$

where¹,

$$D_\mu(\omega)\psi_a := \partial_\mu\psi_a + \frac{1}{2}\omega_{\mu IJ}\sigma^{IJ}\psi_a, \quad D_\mu(\omega)\bar{\psi}_a := \partial_\mu\bar{\psi}_a - \frac{1}{2}\bar{\psi}_a\omega_{\mu IJ}\sigma^{IJ}$$

To this we add the Nieh-Yan density, to write:

$$\mathcal{L} = \frac{1}{2}e\Sigma_{IJ}^{\mu\nu}R_{\mu\nu}{}^{IJ}(\omega) + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\alpha(\omega)\psi_\beta + \frac{\eta}{2}I_{NY}$$

This can be recast as:

$$\mathcal{L} = \frac{1}{2}e\Sigma_{IJ}^{\mu\nu}R_{\mu\nu}^{(\eta)IJ}(\omega) + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\alpha(\omega)\psi_\beta + \frac{\eta}{2}\epsilon^{\mu\nu\alpha\beta}D_\mu(\omega)e_\nu^I D_\alpha(\omega)e_{I\beta} \quad (5)$$

where $R_{\mu\nu}^{(\eta)IJ}(\omega) := R_{\mu\nu}{}^{IJ}(\omega) + \eta\tilde{R}_{\mu\nu}{}^{IJ}(\omega)$

The Nieh-Yan density serves as the term through which η manifests itself as a topological parameter in the supergravity action, and does not show up in the classical equations of motion. This new Lagrangian density also preserves the supersymmetry properties (on-shell) as characterised by (4) since I_{NY} is a total derivative.

Next we develop the analysis in the same manner as done for gravity with spin- $\frac{1}{2}$ fermions in [8]. The 3+1 decomposition of (5) can be achieved through the following parametrisation for the tetrads and their inverses:

$$\begin{aligned} e_t^I &= \sqrt{eN}M^I + N^a V_a^I, \quad e_a^I = V_a^I; \\ M_I V_a^I &= 0, \quad M_I M^I = -1 \\ e_t^t &= -\frac{M_I}{\sqrt{eN}}, \quad e_I^a = V_I^a + \frac{N^a M_I}{\sqrt{eN}}; \\ M^I V_I^a &= 0, \quad V_a^I V_I^b = \delta_a^b, \quad V_a^I V_J^a = \delta_J^I + M^I M_J \end{aligned}$$

Also, we define $q_{ab} := V_a^I V_{bI}$ and $q := \det q_{ab}$ which leads to $e := \det(e_\mu^I) = Nq$.

Ignoring the total spatial derivatives, the Lagrangian density can be written as:

$$\mathcal{L} = e\Sigma_{IJ}^{ta}\partial_t\omega_a^{(\eta)IJ} + t_I^a\partial_t e_a^I - \bar{\pi}^a\partial_t\psi_a - NH - N^a H_a - \frac{1}{2}\omega_t^{IJ}G_{IJ} - 2\bar{S}\psi_t$$

where H , H_a , G_{IJ} and \bar{S} are given below in equation (8) and

$$\begin{aligned} 2e\Sigma_{IJ}^{ta} &= -\sqrt{q}M_{[I}V_{J]}^a \\ t_I^a &= \eta\epsilon^{abc}D_b(\omega)V_{Ic} \\ \bar{\pi}^a &= -\frac{i}{2}\epsilon^{abc}\bar{\psi}_b\gamma_5\gamma_c \end{aligned} \quad (6)$$

¹ The Dirac matrices here obey the Clifford algebra: $\gamma^I\gamma^J + \gamma^J\gamma^I = 2\eta^{IJ}$, $\eta^{IJ} := \text{diag}(-1, 1, 1, 1)$. The chiral matrix $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\sigma^{IJ} := \frac{1}{4}[\gamma^I, \gamma^J]$.

Here $\bar{\pi}^a$ is the canonically conjugate momenta associated with ψ_a ². The last equation in (6) can be inverted as:

$$\bar{\psi}_a = \sqrt{q} \bar{\pi}^b \gamma_a \gamma_b \quad (7)$$

The action does not contain the velocities associated with the gravity fields N, N^a, ω_{tIJ} and the matter field ψ_t . Hence these are Lagrange multipliers, leading to the primary constraints H, H_a, G_{IJ} and \bar{S} , respectively:

$$\begin{aligned} G_{IJ} &= -2D_a(\omega) \left(e\Sigma_{IJ}^{(\eta)ta} \right) - t_{[I}^a V_{J]} a + \bar{\pi}^a \sigma_{IJ} \psi_a \approx 0 \\ H_a &= e\Sigma_{IJ}^{tb} R_{ab}^{(\eta)IJ}(\omega) - V_a^I D_b(\omega) t_I^b + \frac{i}{2} \sqrt{q} \epsilon^{bcd} \bar{\pi}^e \gamma_b \gamma_e \gamma_5 \gamma_a D_c(\omega) \psi_d \approx 0 \\ H &= 2e^2 \Sigma_{IK}^{ta} \Sigma_{JL}^{tb} \eta^{KL} R_{ab}^{(\eta)IJ}(\omega) - \sqrt{q} M^I D_a(\omega) t_I^a + \frac{iq}{2} \epsilon^{abc} \bar{\pi}^d \gamma_a \gamma_d \gamma_5 M_I \gamma^I D_b(\omega) \psi_c \approx 0 \\ \bar{S} &= D_a(\omega) \bar{\pi}^a - \frac{i\sqrt{q}}{4\eta} \bar{\pi}^a \gamma_b \gamma_a \gamma_5 \gamma^I t_I^b \approx 0 \end{aligned} \quad (8)$$

where γ_a is defined as :

$$\gamma_a = \gamma_I V_a^I = (\gamma_i - \gamma_0 \chi_i) V_{ai} \quad (9)$$

While H, H_a, G_{IJ} are the constraints for the pure gravity sector, \bar{S} is the generator of the local supersymmetric transformations.

Following the general framework of ref.[4, 8], we introduce the following set of convenient fields,

$$E_i^a := 2e\Sigma_{0i}^{ta}, \quad \chi_i := -M_i/M^0, \quad A_a^i := \omega_a^{(\eta)0i} - \chi_j \omega_a^{(\eta)ij}, \quad \zeta^i := -E_j^a \omega_a^{(\eta)ij} \quad (10)$$

alongwith the decomposition of the nine components of $\omega_a^{(\eta)ij}$ in terms of three ζ_i 's and six M_{kl} 's ($M_{kl} = M_{lk}$) :

$$\omega_a^{(\eta)ij} = \frac{1}{2} (E_a^{[i} \zeta^{j]} + \epsilon^{ijk} E_{al} M^{kl}) \quad (11)$$

In terms of the fields in (10), we have $2e\Sigma_{ij}^{ta} = -E_{[i}^a \chi_{j]}$ and $e\Sigma_{IJ}^{ta} \partial_t \omega_a^{(\eta)IJ} = E_i^a \partial_t A_a^i + \zeta^i \partial_t \chi^i$. Note that the eighteen coordinate variables ω_a^{IJ} have been reexpressed in terms of the twelve variables A_{ai} and χ_i . The remaining six variables are the M_{kl} 's, whose velocities do not appear in the Lagrangian density. Hence these are the additional Lagrange multiplier fields.

² The functional derivative involving the Grassmann variables (fermions) acts on the left factor resulting in a sign in the definition of the conjugate momenta in (6).

Thus the Lagrangian density takes a simple form as follows:

$$\mathcal{L} := E_i^a \partial_t A_a^i + \zeta^i \partial_t \chi^i + t_I^a \partial_t V_a^I - \bar{\pi}^a \partial_t \psi_a - NH - N^a H_a - \frac{1}{2} \omega_t^{IJ} G_{IJ} - 2\bar{S} \psi_t$$

The fields V_a^I and t_I^a are not really independent, these are given in terms of the basic fields as: $V_a^I = v_a^I$ and $t_I^a = \tau_I^a$ where

$$\begin{aligned} v_a^0 &:= -\frac{1}{\sqrt{E}} E_a^i \chi_i, \quad v_a^i := \frac{1}{\sqrt{E}} E_a^i \\ \tau_0^a &:= \eta \epsilon^{abc} D_b(\omega) v_{0c} \\ &= \eta \sqrt{E} E_m^a \left[G_{\text{rot}}^m - \frac{\chi_l}{2} \left(\frac{2f_{ml} + N_{ml}}{1 + \eta^2} + \epsilon_{mln} G_{\text{boost}}^n \right) - i\bar{\pi}^b \gamma_5 (\sigma_{0m} + \frac{\chi_l}{2} \sigma_{ml}) \psi_b \right] \\ \tau_k^a &:= \eta \epsilon^{abc} D_b(\omega) v_{ck} \\ &= -\frac{\eta}{2} \sqrt{E} E_m^a \left[\frac{2f_{mk} + N_{mk}}{1 + \eta^2} + \epsilon_{kmn} G_{\text{boost}}^n + i\bar{\pi}^b \gamma_5 \sigma_{km} \psi_b \right] \end{aligned} \quad (12)$$

In the above, f_{kl} and N_{kl} are defined as:

$$2f_{kl} := \epsilon_{ijk} E_i^a [(1 + \eta^2) E_b^l \partial_a E_j^b + \chi_j A_a^l] + \eta (E_l^a A_a^k - \delta^{kl} E_m^a A_a^m - \chi_l \zeta_k) + (l \leftrightarrow k) \quad (13)$$

$$N_{kl} := (\chi^2 - 1)(M_{kl} - M_{mm} \delta_{kl}) + \chi_m \chi_n M_{mn} \delta_{kl} + \chi_l \chi_k M_{mm} - \chi_m (\chi_k M_{ml} + \chi_l M_{mk}) \quad (14)$$

We shall treat V_a^I and t_I^a as independent variables and introduce associated Lagrange multipliers ξ_I^a and ϕ_a^I to express the equations in (12) as constraints.

Thus we write the full Lagrangian density as,

$$\begin{aligned} \mathcal{L} = & E_i^a \partial_t A_a^i + \zeta^i \partial_t \chi^i + t_I^a \partial_t V_a^I - \bar{\pi}^a \partial_t \psi_a - NH - N^a H_a - \frac{1}{2} \omega_t^{IJ} G_{IJ} \\ & - \xi_I^a (V_a^I - v_a^I) - \phi_a^I (t_I^a - \tau_I^a) - 2\bar{S} \psi_t \end{aligned} \quad (15)$$

The constraints in (8) can now be rewritten in terms of the canonical fields. These can be worked out in an analogous manner as in ref.[8]. Thus the corresponding expressions for $G_i^{\text{boost}} := G_{0i}$, $G_i^{\text{rot}} := \frac{1}{2} \epsilon^{ijk} G_{jk}$, H_a , H and \bar{S} are:

$$\begin{aligned} G_{\text{boost}}^i &= -\partial_a (E_i^a - \eta \epsilon^{ijk} E_j^a \chi_k) + E_{[i}^a \chi_{k]} A_a^k + (\zeta^i - \chi \cdot \zeta \chi^i) - t_{[0}^a V_{i]a} \\ &\quad + \bar{\pi}^a \sigma_{0i} \psi_a + \frac{\eta}{4M^0 E} \epsilon^{ijk} E_{al} E_k^b \bar{\pi}^a (\gamma_j - \gamma_0 \chi_j) (\gamma_l - \gamma_0 \chi_l) (\gamma_0 - \gamma_m \chi_m) \psi_b; \\ G_{\text{rot}}^i &= \partial_a (\epsilon^{ijk} E_j^a \chi_k + \eta E_i^a) + \epsilon^{ijk} (A_a^j E_k^a - \zeta_j \chi_k - t_j^a V_a^k) \\ &\quad + i\bar{\pi}^a \gamma_5 \sigma_{0i} \psi_a - \frac{\eta}{4M^0 E} E_{al} \bar{\pi}^a (\gamma_{[i} - \gamma_0 \chi_{i]} E_{j]}^b (\gamma_l - \gamma_0 \chi_l) \gamma_j \psi_b; \\ H_a &= E_i^b \partial_{[a} A_{b]}^i + \zeta_i \partial_a \chi_i - \partial_b (t_I^b V_a^I) + t_I^b \partial_a V_b^I \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{1+\eta^2} [E_{[i}^b \chi_{l]} A_b^l + \zeta_i - \chi \cdot \zeta \chi^i - t_{[0}^b V_{i]b} - \eta \epsilon^{ijk} (A_a^j E_k^b + \chi_j \zeta_k - t_j^b V_b^k)] A_a^i \\
& -\frac{1}{1+\eta^2} \left[\frac{1}{2} \epsilon^{ijk} (\eta G_{\text{boost}}^k + G_{\text{rot}}^k) - \chi^i (G_{\text{boost}}^j - \eta G_{\text{rot}}^j) \right] \omega_a^{(\eta)ij} \\
& -\frac{1}{4(1+\eta^2)} \frac{1}{M^0 \sqrt{E}} \epsilon^{bcd} \bar{\pi}^e \gamma_b \gamma_e (\eta - i\gamma_5) \gamma_{[a} \omega_{c]}^{(\eta)ij} (\sigma_{ij} + 2\sigma_{0i} \chi_j) \psi_d \\
& -\frac{1}{2(1+\eta^2)} \frac{1}{M^0 \sqrt{E}} \epsilon^{bcd} \bar{\pi}^e \gamma_b \gamma_e \gamma_a (\eta + i\gamma_5) \sigma_{0i} A_c^i \psi_d \\
H = & -E_k^a \chi_k H_a + (1 - \chi \cdot \chi) \left[E_i^a \partial_a \zeta_i + \frac{1}{2} \zeta_i E_i^a E_j^b \partial_a E_b^j \right] \\
& + \frac{1 - \chi \cdot \chi}{2(1+\eta^2)} \zeta_i [-G_{\text{boost}}^i + \eta G_{\text{rot}}^i] - (E_k^a \chi_k V_a^I + \sqrt{q} M^I) \partial_b t_I^b \\
& -\frac{1 - \chi \cdot \chi}{1 + \eta^2} \left[\frac{1}{2} E_{[i}^a E_{j]}^b A_a^i A_b^j + E_i^a A_a^i \chi \cdot \zeta + \eta \epsilon^{ijk} \zeta_i A_a^j E_k^a \right. \\
& \quad \left. + \frac{3}{4} (\chi \cdot \zeta)^2 - \frac{3}{4} (\zeta \cdot \zeta) + \frac{1}{2} \zeta_i t_{[0}^a V_{i]a} - \frac{\eta}{2} \zeta_i \epsilon^{ijk} t_j^a V_a^k \right] \\
& + \frac{1 - \chi \cdot \chi}{1 + \eta^2} \left[\frac{1}{\sqrt{E}} A_a^i t_i^a + \frac{1}{2} V_a^i (\zeta \cdot \chi t_i^a - \chi_i \zeta_j t_j^a + \eta \epsilon^{ijk} \zeta_j t_k^a) \right] \\
& + \frac{1 - \chi \cdot \chi}{2(1+\eta^2)} \left[(\eta t_k'^a - \epsilon^{ijk} \chi_i t_j'^a) \frac{E_{al}}{\sqrt{E}} + 2f_{kl} + \frac{1}{2} N_{kl}(M) + 2(1+\eta^2) J_{kl} \right] M^{kl} \\
& -\frac{1 - \chi \cdot \chi}{2(1+\eta^2)} \frac{1}{M^0 E} \epsilon^{abc} \bar{\pi}^d \gamma_a \gamma_d \gamma_0 (\eta + i\gamma_5) \left(\sigma_{0i} A_b^i \psi_c + \frac{1}{4} (\sigma_{ij} + 2\sigma_{0i} \chi_j) E_{b[i} \zeta_{j]} \psi_c \right) \\
& + \frac{1 - \chi \cdot \chi}{2(1+\eta^2)} \zeta_i \bar{\pi}^a (1 - i\eta \gamma_5) \sigma_{0i} \psi_a \\
\bar{S} = & \partial_a \bar{\pi}^a - \frac{1}{1+\eta^2} \bar{\pi}^a (1 - i\eta \gamma_5) \left[\sigma_{0l} A_a^l + \frac{1}{2} (\sigma_{ij} + 2\sigma_{0i} \chi_j) \omega_{aij}^{(\eta)} \right] - \frac{i}{4\eta M^0 \sqrt{E}} \bar{\pi}^a \gamma_b \gamma_a \gamma_5 \gamma^I t_I^b
\end{aligned} \tag{16}$$

where we have used the definitions:

$$\begin{aligned}
t_I'^a &:= t_I^a - \frac{\eta}{4} \epsilon^{abc} \bar{\psi}_b \gamma_I \psi_c \\
&= t_I^a - \frac{\eta}{4M^0 \sqrt{E}} \epsilon^{ijk} E_k^a E_j^c E_b^l \bar{\pi}^b (\gamma_i - \gamma_0 \chi_i) (\gamma_l - \gamma_0 \chi_l) \gamma_I \psi_c
\end{aligned} \tag{17}$$

$$\begin{aligned}
2J_{kl} &:= \frac{1}{4} \epsilon^{abc} \bar{\psi}_b \gamma_k \psi_c E_{al} + (k \leftrightarrow l) \\
&= \frac{1}{4M^0 \sqrt{E}} \epsilon^{iml} E_{aj} E_m^b \bar{\pi}^a (\gamma_i - \gamma_0 \chi_i) (\gamma_j - \gamma_0 \chi_j) \gamma_k \psi_b + (k \leftrightarrow l)
\end{aligned} \tag{18}$$

The Hamiltonian density now reads:

$$\mathcal{H} = NH + N^a H_a + \frac{1}{2} \omega_t^{IJ} G_{IJ} + \xi_I^a (V_a^I - v_a^I) + \phi_a^I (t_I^a - \tau_I^a) + 2\bar{S} \psi_t$$

The constraints associated with the fields $N^a, N, \omega_t^{0i}, \omega_t^{ij}, \xi_I^a, \phi_a^I$ and ψ_t respectively are:

$$\begin{aligned} H_a &\approx 0 \quad , \quad H \approx 0 \quad , \quad G_{\text{boost}}^i \approx 0 \quad , \quad G_{\text{rot}}^i \approx 0 \\ V_a^I - v_a^I &\approx 0 \quad , \quad t_I^a - \tau_I^a \approx 0 \quad , \quad \bar{S} \approx 0. \end{aligned}$$

As mentioned earlier, the momenta conjugate to M_{kl} are zero. The preservation of this constraint requires:

$$\frac{\delta H}{\delta M_{kl}} \approx 0 \quad ,$$

which implies:

$$(\eta t_k'^a - \epsilon^{ijk} \chi_i t_j'^a) V_a^l + f_{kl} + \frac{1}{2} N_{kl} + (1 + \eta^2) J_{kl} + (k \leftrightarrow l) \approx 0 \quad (19)$$

where, f_{kl} and N_{kl} are given in (13, 14). This constraint can be solved for M_{kl} . Next, using $t_I^a \approx \tau_I^a$, we write

$$\begin{aligned} t_k'^a &\approx -\frac{\eta}{2} \sqrt{E} E_l^a \left[\frac{2f_{kl} + N_{kl}}{1 + \eta^2} + 2J_{kl} + \epsilon_{kl n} G_{\text{boost}}'^n \right] \\ t_0'^a &\approx \eta \sqrt{E} E_l^a \left[G_{\text{rot}}'^l - \frac{\chi_k}{2} \left(\frac{2f_{kl} + N_{kl}}{1 + \eta^2} + 2J_{kl} + \epsilon_{kl n} G_{\text{boost}}'^n \right) \right] \end{aligned} \quad (20)$$

Using (20) in (19), we obtain

$$2f_{kl} + N_{kl} + 2(1 + \eta^2) J_{kl} \approx 0 \quad (21)$$

Thus the J_{kl} piece captures all the contribution coming from the spin- $\frac{3}{2}$ fermions. Note that this equation has the same form as the one for spin- $\frac{1}{2}$ fermions [8]. This constraint, from (20), further implies:

$$t_I'^a \approx 0 \quad (22)$$

This is exactly same as the connection equation of motion which is obtained in the Lagrangian formulation by varying the standard supergravity action without the Nieh-Yan term (see [6], for example).

Using (22), the final set of constraints read:

$$\begin{aligned} G_{\text{boost}}^i &= -\partial_a (E_i^a - \eta \epsilon^{ijk} E_j^a \chi_k) + E_{[i}^a \chi_{k]} A_a^k + (\zeta^i - \chi \cdot \zeta \chi^i) \\ &\quad + \bar{\pi}^a \sigma_{0i} \psi_a + \frac{\eta}{4M^0 E} \epsilon^{ijk} E_{al} E_k^b \bar{\pi}^a (\gamma_j - \gamma_0 \chi_j) (\gamma_l - \gamma_0 \chi_l) (\gamma_0 - \gamma_m \chi_m) \psi_b \\ G_{\text{rot}}^i &= \partial_a (\epsilon^{ijk} E_j^a \chi_k + \eta E_i^a) + \epsilon^{ijk} (A_a^j E_k^a - \zeta_j \chi_k) + i \bar{\pi}^a \gamma_5 \sigma_{0i} \psi_a \end{aligned}$$

$$\begin{aligned}
& - \frac{\eta}{4M^0 E} E_{al} \bar{\pi}^a (\gamma_{[i} - \gamma_0 \chi_{[i}) E_{j]}^b (\gamma_l - \gamma_0 \chi_l) \gamma_j \psi_b \\
H_a = & E_i^b \partial_{[a} A_{b]}^i + \zeta_i \partial_a \chi_i - \partial_b \left((\tau_i'^b - \chi_i \tau_0'^b) \frac{E_a^i}{\sqrt{E}} \right) - \tau_0'^b \partial_a \left(\chi_i \frac{E_b^i}{\sqrt{E}} \right) + \tau_i'^b \partial_a \left(\frac{E_b^i}{\sqrt{E}} \right) \\
& - \frac{1}{1+\eta^2} \left[E_{[i}^b \chi_{l]} A_b^l + \zeta_i - \chi \cdot \zeta \chi^i - \frac{1}{\sqrt{E}} \tau_{[0}^b E_{i]b} - \eta \epsilon^{ijk} (A_a^j E_k^b + \chi_j \zeta_k - \frac{1}{\sqrt{E}} \tau_j'^b E_k^b) \right] A_a^i \\
& - \frac{1}{8(1+\eta^2)} \frac{1}{M^0 \sqrt{E}} \epsilon^{bcd} \bar{\pi}^e \gamma_b \gamma_e (\eta - i\gamma_5) \gamma_{[a} (E_{c]}^{[i} \zeta^{j]} + \epsilon^{ijm} E_{c]}^n M_{mn}) (\sigma_{ij} + 2\sigma_{0i} \chi_j) \psi_d \\
& - \frac{1}{2(1+\eta^2) M^0 \sqrt{E}} \epsilon^{bcd} \bar{\pi}^e \gamma_b \gamma_e (\eta - i\gamma_5) \gamma_a \sigma_{0k} A_c^k \psi_d \\
H = & (1 - \chi \cdot \chi) \left[E_i^a \partial_a \zeta_i + \frac{1}{2} \zeta_i E_i^a E_j^b \partial_a E_b^j \right] - \frac{1 - \chi \cdot \chi}{\sqrt{E}} \partial_b \tau_0'^b \\
& - \frac{1 - \chi \cdot \chi}{1 + \eta^2} \left[\frac{1}{2} E_{[i}^a E_{j]}^b A_a^i A_b^j + E_i^a A_a^i \chi \cdot \zeta + \eta \epsilon^{ijk} \zeta_i A_a^j E_k^a \right. \\
& \quad \left. + \frac{3}{4} (\chi \cdot \zeta)^2 - \frac{3}{4} (\zeta \cdot \zeta) + \frac{1}{2\sqrt{E}} \zeta_i (\tau_0'^a - \chi_k \tau_k'^a) E_a^i - \frac{\eta}{2\sqrt{E}} \zeta_i \epsilon^{ijk} \tau_j'^a E_k^a \right] \\
& + \frac{1 - \chi \cdot \chi}{1 + \eta^2} \left[\frac{1}{\sqrt{E}} A_a^i \tau_i'^a + \frac{1}{2\sqrt{E}} E_a^i (\zeta \cdot \chi \tau_i'^a - \chi_i \zeta_j \tau_j'^a + \eta \epsilon^{ijk} \zeta_j \tau_k'^a) \right] \\
& - \frac{1 - \chi \cdot \chi}{2(1 + \eta^2)} \frac{1}{M^0 E} \epsilon^{abc} \bar{\pi}^d \gamma_a \gamma_d \gamma_0 (\eta + i\gamma_5) \left(\sigma_{0i} A_b^i \psi_c + \frac{1}{4} (\sigma_{ij} + 2\sigma_{0i} \chi_j) E_{b[i} \zeta_{j]} \psi_c \right) \\
& + \frac{1 - \chi \cdot \chi}{2(1 + \eta^2)} \zeta_i \bar{\pi}^a (1 - i\eta \gamma_5) \sigma_{0i} \psi_a + \frac{1 - \chi^2}{4(1 + \eta^2)} [f_{kl} + (1 + \eta^2) J_{kl}] M^{kl} \\
\bar{S} = & \partial_a \bar{\pi}^a - \frac{1}{1 + \eta^2} \bar{\pi}^a (1 - i\eta \gamma_5) \left[\sigma_{0l} A_a^l + \frac{1}{4} (\sigma_{ij} + 2\sigma_{0i} \chi_j) (E_{a[i} \zeta_{j]} + \epsilon_{ijl} E_{am} M^{lm}) \right]
\end{aligned}$$

where $\tau_I'^a$ is defined as

$$\begin{aligned}
\tau_I'^a &:= \frac{\eta}{4} \epsilon^{abc} \bar{\psi}_b \gamma_I \psi_c \\
&= \frac{\eta}{4M^0 \sqrt{E}} \epsilon^{ijk} E_k^a E_j^c E_b^l \bar{\pi}^b (\gamma_i - \gamma_0 \chi_i) (\gamma_l - \gamma_0 \chi_l) \gamma_I \psi_c
\end{aligned} \tag{23}$$

and f_{kl} , J_{kl} and M_{kl} are given by the (13), (14), (18) and (21). In writing \bar{S} , we have made use of the Fierz identity-

$$\epsilon^{\mu\nu\alpha\beta} \bar{\psi}_\mu \gamma_I \psi_\nu \gamma^I \psi_\alpha = 0,$$

which makes the piece proportional to t_I^a dissappear.

Time gauge:

One may adopt the time gauge through the choice $\chi_i = 0$. Since this condition forms a second-class pair with the boost constraint, both have to be implemented together. G_i^{boost}

can be solved as:

$$\zeta_i = \partial_a E_i^a - \bar{\pi}^a \sigma_{0i} \psi_a - \frac{\eta}{4M^0 E} \epsilon^{ijk} E_{al} E_k^b \bar{\pi}^a \gamma_j \gamma_l \gamma_0 \psi_b \quad (24)$$

We can rewrite this as:

$$\zeta_i = \partial_a E_i^a + \frac{1}{\sqrt{E}} \tau_0'^a E_{ai} + \frac{1}{\eta \sqrt{E}} \epsilon^{ijk} \tau_j'^a E_{ak} \quad (25)$$

with

$$\tau_I'^a = \frac{\eta}{4\sqrt{E}} \epsilon^{ijk} E_k^a E_j^c E_{bl} \bar{\pi}^b \gamma_i \gamma_l \gamma_I \psi_c$$

The constraints in this gauge read:

$$\begin{aligned} G_{\text{rot}}^i &= \eta \partial_a E_i^a + \epsilon^{ijk} A_a^j E_k^a - \frac{1}{\eta \sqrt{E}} \tau_0'^a E_{ai} - \frac{1}{\sqrt{E}} \epsilon^{ijk} \tau_j'^a E_{ak} \\ H_a &= E_i^b F_{ab}^i - \frac{1}{\eta^2 \sqrt{E}} \tau_0'^b E_{bi} A_{ai} - \frac{E_{ai}}{\sqrt{E}} \left[\partial_b \tau_i'^b + \frac{1}{\eta} \epsilon^{ijk} A_b^j \tau_k'^b \right] + \frac{1}{2(1+\eta^2)} [A_a^k - E_a^i E_k^b A_b^i] \zeta_k \\ H &= -\frac{\eta}{2} E_i^a E_j^b \epsilon^{ijk} \left[F_{ab}^k + \left(\eta + \frac{1}{\eta} \right) R_{ab}^k \right] - \frac{1}{\sqrt{E}} \left[\partial_a \tau_0'^a - \frac{1}{2\eta} (\epsilon^{ijk} E_a^j \zeta_k \tau_i'^a + E_a^j \tau_i'^a M^{ij}) \right] \\ \bar{S} &= \partial_a \bar{\pi}^a - \frac{1}{1+\eta^2} \bar{\pi}^a (1 - i\eta \gamma_5) \sigma_{0k} \left[A_a^k + \frac{1}{2} i\gamma_5 (\epsilon_{jkl} \zeta_j + M^{kl}) E_{al} \right] \end{aligned} \quad (26)$$

In these equations, we have used the following definitions:

$$\begin{aligned} \Gamma_{ai} &= \frac{1}{2} \epsilon^{ijk} \omega_{ajk} \\ F_{ab}^k &= \partial_{[a} A_{b]}^k + \frac{1}{\eta} \epsilon^{ijk} A_{ai} A_{bj} \quad , \quad R_{ab}^k = \partial_{[a} \Gamma_{b]}^k - \frac{1}{\eta} \epsilon^{ijk} \Gamma_{ai} \Gamma_{bj} \end{aligned}$$

and ζ^i is given by (25). Also, in the time gauge :

$$\begin{aligned} M_{kl} &= (1+\eta^2) (\epsilon^{ijk} E_b^l \partial_a E_j^b - \epsilon^{ijm} E_b^m \partial_a E_j^b \delta_{kl}) E_i^a + (1+\eta^2) (2J_{kl} - J_{mm} \delta_{kl}) \\ &\quad + \eta E_l^a A_{ak} + (k \leftrightarrow l) \\ 2J_{kl} &= \frac{1}{4M^0 \sqrt{E}} \epsilon^{iml} E_{aj} E_m^b \bar{\pi}^a \gamma_i \gamma_j \gamma_k \psi_b + (k \leftrightarrow l) \end{aligned}$$

Here in (26) we have dropped terms proportional to rotation constraints from H_a and H .

As is evident, the dynamical variable which enters in the constraints apart from the fermionic degrees of freedom is the Barbero-Immirzi connection A_a^i . Thus in the time gauge we obtain a real SU(2) formulation of the theory of gravity coupled to spin- $\frac{3}{2}$ fermions.

Notice that in the matter sector, $\bar{\pi}^a$ and ψ_a are not independent variables. These obey the second-class constraints:

$$C^a := \bar{\pi}^a + \frac{i}{2} \epsilon^{abc} \bar{\psi}_b \gamma_5 \gamma_c \approx 0$$

In order to implement these constraints, we need to go to corresponding Dirac brackets for the matter fields $\bar{\pi}^a, \psi_a$. This then leads to the correct transformations (modulo rotations) on the fields through their Dirac brackets with the corresponding generators. In particular, the Dirac brackets of the fields with the supersymmetry generator \bar{S} make them transform properly under its action.

III. CONCLUSIONS

We have presented a framework to incorporate the Barbero-Immirzi parameter as a topological coupling constant in the classical theory of $N = 1$ supergravity. This is achieved through the inclusion of the Nieh-Yan density in the Lagrangian. This additional term, being a topological density, preserves the equations of motion and the supersymmetry of the original action. To emphasise, this goes beyond the earlier analysis involving the Holst action which does not allow a topological interpretation for η .

The canonical formulation has been first developed without going to any particular choice of gauge. This clarifies the structure of the theory exhibiting all of its gauge freedom. In the time gauge, the theory is shown to admit a real $SU(2)$ formulation in terms of the Barbero-Immirzi connection A_a^i .

The essential features for spin- $\frac{3}{2}$ fermions turn out to be very similar to those for spin- $\frac{1}{2}$ fermions as described in [8], except that here we have the additional constraint \bar{S} which acts as the generator of local supersymmetry transformations. The cases for $N = 2, 4$ and higher supergravity theories can be treated in exactly similar fashion. There the constraint analysis leads to the same form of the connection equation of motion as given here (i.e., equation (21)), a fact which is evident from the structure of the fermionic terms in these theories. Only the expression for J_{kl} in terms of the matter fields gets modified.

The analysis here has been purely classical. However, in the quantum theory, the presence of the topological Nieh-Yan term, which is also CP violating, may reflect a possible non-perturbative vacuum structure.

Acknowledgements:

We thank Prof. Ghanashyam Date and V. S. Nemani for useful discussions.

- [1] Amitabha Sen, Phys. Lett. **B 119** (1982) 89-91 ;
A. Ashtekar, Phys. Rev. Lett. **57** (1986) 2244-2247;
A. Ashtekar, Phys. Rev. **D36** (1987) 1587-1602
- [2] J. Fernando G. Barbero, Phys. Rev. **D51** (1995) 5507-5510;
Giorgio Immirzi, Class. Quant. Grav. **14** (1997) L177-L181
- [3] S. Holst, Phys. Rev. **D53** (1996) 5966-5969
- [4] Nuno Barros e Sa, Int. J. Mod. Phys. **D10** (2001) 261-272
S. Alexandrov, Class. Quantum Grav. **17** (2000) 4255-4268
- [5] Simone Mercuri, Phys. Rev. **D73** (2006) 084016, [gr-qc/0601063];
Simone Mercuri, [arXiv:gr-qc/0610026]
- [6] Romesh K. Kaul, Phys. Rev. **D77** (2008) 045030
- [7] S. James Gates Jr., Sergei V. Ketov, Nicolas Yunes, [arXiv:hep-th/0906.4978]
- [8] G. Date, R. K. Kaul, S. Sengupta, Phys. Rev. **D79** (2009) 044008
- [9] H. T. Nieh and M. L. Yan, J. Math. Phys. **23** (1982) 373-374;
H. T. Nieh, Int. J. Mod. Phys., **A 22** (2007) 5237-5244
- [10] Pilati, Nucl. Phys. **B 132** (1978)
- [11] M. Tsuda, Phys. Rev. **D61** (2000) 024025 ;
M. Sawaguchi, Class. Quantum Grav. **18** (2001) 1179
- [12] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, Phys. Rev. **D13** (1976) 3214;
S. Deser and B. Zumino, Phys. Lett. **62B** (1976) 335;
D. Z. Freedman and P. van Nieuwenhuizen, Phys. Rev. **D14** (1976) 912